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On the Ingredients of the Twin Paradox

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Abstract If M is a 4-dimensional connected, orientable flat spacetime manifold endowed with a time-arrow, and if the existence of a finite speed limit to energy/information transfers over M can be assessed, then the "twin paradox" necessarily follows (and indeed, the full special relativity theory). Two other implicit ingredients of the paradox are also identified.

Keywords Twin paradox · Minkowski spacetime · Causality · Finite speed limit

1 Introduction

As W.G. Unruh wrote more than 20 years ago, "the twin paradox has generated one of the longest standing controversies in twentieth century physics" [27]. Of course, there is a big deal of literature on the subject, and it seems that publications are still going on.

Indeed, if the mathematical concept of a (pseudo-) Riemannian manifold is adequate to the description of *History*, envisaged as the set of all spacetime events, then, it must definitely be stated that any sort of Langevin's twin paradox gets substantially re-interpreted. This is so because of a one, long known fact. With Sommerfeld's own words, "As Minkowski once remarked to me, the element of *proper-time* is not an exact differential" [23]. Proper-time lapses are therefore worldlines functionals. This statement ruins the possible bases of any twin paradox, at least as understood originally, and is a general, geometric property of Riemannian and, to some extend, of pseudo-Riemannian manifolds.

For compact Riemannian manifolds, in effect, the Hopf-Rinow theorem ensures that $\mathcal{M}$ is also *geodesically complete*, and that any pair of points can be joined by a geodesic of minimal length. If two points are separated enough, they may be joined by another geodesic whose length can differ from the minimal one. Differential ageing along intersecting worldlines will therefore arise as an intrinsic geometrical property of the Riemannian manifold

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itself, whatever the worldlines. In the physically interesting case of compact or so-called "*t-complete*" Lorentzian manifolds, geodesic completeness has been shown to extend to the timelike and lightlike cases, that are relevant to the twin paradox [4].

This could be the end of the story. However, one may consider a geometrical description as a final, *effective* description which, when achieved, has erased the physical mechanisms at its origin, providing us with the result, geometrical, of all the *forces* having shaped it. A striking illustration of this claim can be learned out of the approach outlined in [1]. That is, however elegant and satisfying in some respects, a geometrical description may hide some fundamental physical principle at the origin of a given phenomenon.

In this article, our intention is to *unfold* somewhat the kind of geometrical description alluded to above, so as to pin up the mechanism and/or principle responsible for the non-trivial differential ageing phenomenon. One may think, in effect, that present days technologies have shed new and decisive lights on questions that pure speculations revealed unable to fully elucidate. And it is conceivable that here is also one of the most compelling reason at the origin of a so long and vivid controversy. After all, "Science, has the same age as its technology", Wisdom says.

A rapid review of the twin paradox is given in Sect. 2. To begin with, Sect. 3 considers a confusion at the origin of the famous paradox; in order to preserve a long used terminology, the paradox appellation is re-defined. Then, for the sake of a later geometrical synthesis, the closely related Thomas-Wigner rotations phenomenon is introduced. Eventually the definition of the proper-time lapse functional is given. Within elementary differential geometry settings, the matter of Sect. 4 is the relation of the re-qualified twin paradox to the principle of causality, and Sect. 5 concludes the article.

2 Paradox Settings

As is well known, the famous paradox lies in its *reciprocity*. For either twin, in effect, non-trivial differential ageing should manifest itself in exactly the same way, an obvious contradiction if the phenomenon is to be real. That is, the matter of *reciprocity* is to be examined with the one of *reality*.

Is this a sound consideration? A glance at History, even recent enough, should let no doubt about the answer [3]. Major physicists, as well as philosophers, have long thought of relativistic effects in terms of *appearances*, to fade away as soon as *real* comparisons of twins or clocks are duly achieved [6, 10]. Indeed, nothing in the special relativity formalism could prevent them to think so, quite on the contrary. From the onset, in effect, clocks and rods are assumed to be identical, in all of the inertial frames of reference (see [23], for example, pp. 72, 82 and 88, note 13).

In this respect, one may notice that even recent enough terminologies, like the one of *parallax effects* for *time dilatation factors*, for example, entail that connotation of *appearances*, though in a context where the relation *real/apparent* is clearly exposed [15].

It would seem that the issue of *reality* should be disposed of easily. In effect, if special relativity parallax effects are nothing but pure appearances, then, the latter could simply fade away at the worldlines intersecting points: H. Bergson and H. Dingle are right (to make a long story short!), there is neither any real differential ageing, nor any paradox whatsoever. If on the contrary the so-called parallax effects are real, in the sense of persistent and actually measurable, then differential ageing is real in the same acceptation, and an established *reciprocity* (symmetry) or *non-reciprocity* (asymmetry) has to be explained.

At first, it is important to point out a confusion at the origin of the idea of a symmetrical and paradoxical case.

3 Parallax and Paradox

3.1 A Historical Avatar

It is worth recalling how the idea of a paradox emerged and developed. As recently put forth in much details [23], standard special relativity formalisms are mixtures of Einstein's kinematics and Poincaré's group theory, that is, the group of *scalar* boosts in a given direction, which is a commutative one dimensional group.

This remark elucidates a part of the twin controversy, because it has long reduced the paradox discussions to an *effective* 1 time + 1 space dimensional case. There, as compared to the 1 time + 2 or 3 space dimension case to be discussed, the *inertial frames reciprocity* is so naturally preserved, that it is not easy to figure out how an asymmetrical twin situation could possibly show up.

By *inertial frames reciprocity*, the following is meant. For any two given inertial frames of reference, $K(v)$ and $K'(v')$ (a third inertial laboratory frame, K_0, being understood with respect to which v and v' make sense), $K'(v')$ is seen from $K(v)$ the same, still opposite way, as $K(v)$ is seen from $K'(v')$, in agreement with the postulated equivalence of inertial frames [2, 11].

Literature keeps track of that difficulty, which H. Bergson and H. Dingle, for example, could not think of another way than by appealing to *appearances*. A part of truth was indeed contained in their point of view, the clue being that instead of a genuine paradoxical situation, to be re-defined shortly, the situation here is rather that of a parallax effect. Parallax effects are not Lorentz invariants, and may be thought of as *appearances*. But then, an important *proviso* should be made: these appearances give rise to persistent, really measurable effects, just like their 3-euclidean space homologous do.

Facts observed long ago in elementary particles accelerators are relevant to this situation. The textbook example of π-meson beams can help fixing the idea and the terminology. At a speed close to c, the π-mesons travel over distances corresponding to *life-durations* which can be a hundred times longer than their lifetimes of 10^{-8} s. This is a physically measurable effect, and a persistent one since it is even used to keep the π-meson factory away from the experimental zone.

However, there is nothing more here than a pure parallax effect, due to the fact that the 1 time + 1 space laboratory axes are *hyperbolically* rotated with respect to the inertial beam spatio-temporal axes: the laboratory *life-duration* measurement is not performed in the π-meson beam *proper-frame*, and provides the real measurement of a real parallax effect [16, 23]. Under the exchange of inertial reference frames, the perfect symmetry of this parallax effect has led to the idea of a paradox when clocks (twins) had both to run fast with respect to the other!

Now, concerning Lorentz invariant quantities, there is no paradox nor any asymmetry. Proper-time lapses are the same in either cases, and a beam-embarked clock would deliver the same indication as a laboratory one, an averaged lifetime of 10^{-8} s (the development to come may suggest that there is more to say about this example which is reduced here to a somewhat *academic* form. This form however, is sufficient to our immediate purpose)

Though sometimes misleading, Science does not often revoke its original terminologies, and we will keep using the word of "paradox". In the present case, however, it seems appropriate to reserve the paradoxical epithet for those situations involving invariant quantities only. In this way, any risk of confusion with parallax effects is avoided, right from the onset. In the more elaborate case of general relativity, a twin paradoxical situation will be translated into the terms of the present article's introduction: two point-events of a given spacetime manifold can be joined by arcs corresponding to different proper-time intervals ... a still rather counter-intuitive fact indeed, and a one already present at the more elementary scale of Minkowski space geometry, as we will see.

3.2 A Closely Related and Remarkable Mechanism

Other specific relativistic effects are worth of attention in themselves, as well as in their relation to the twin paradox. Since only certain quadratic combinations of them form invariant quantities, relativistic theories do not really discriminate spatial from temporal coordinates. This is clearly read off a most classical transformation *formulae* like

$$x' = \gamma(x - vt), \qquad y' = y, \qquad z' = z, \qquad ct' = \gamma\left(ct - \frac{vx}{c}\right) \tag{1}$$

At the level of spatial coordinates, this symmetry could have helped anticipating the existence of the non trivial behaviours that are known presently under the spell of *Thomas-Wigner rotations*.

At one more spatial dimensions in effect (2 instead of 1), the situation begins to change in many respects. Considered from the laboratory frame K_0, two successive boosts, $K_0 \to K(\vec{v}) \to K'(\vec{v'})$, may have non-collinear velocity vectors, and then, the 1-space unescapable *inertial frames reciprocity* looks jeopardized because from K_0, $K'(\vec{v'})$ is no longer seen the same, still opposite way, as K_0 is seen from $K'(\vec{v'})$!

With the axes of $K(\vec{v})$ taken parallel to those of K_0, and the axes of $K'(\vec{v'})$, parallel to those of $K(\vec{v})$, it is assumed that $\vec{v}$ is along the Ox direction of K_0, and $\vec{v'} = d\vec{v}$, along the Oy direction of $K(\vec{v})$. Then, the K_0 versus K' relative velocity, expressed in K' and in K_0, points to different directions. An expression like

$$\mathrm{d}\theta = \frac{|\mathrm{d}\vec{v}_{Oy}|}{|\vec{v}_{Ox}|}\left(1 - \sqrt{1 - \frac{\vec{v}_{Ox}^2}{c^2}}\right) \tag{2}$$

is a typical textbook equation, which to first order, accounts for such an angular difference [2, 11]. This shows that contrarily to the non-relativistic case, inertial frames parallelism is not a transitive relation in special relativity theory.

In view of this, the simplest way to recover an essential inertial frames reciprocity, consists in stating that seen from K_0, the spatial axes of $K'(\vec{v'})$ have *rotated* some angle $\mathrm{d}\theta$ [16].

Now, an important point is that the Thomas-Wigner rotation also defines a fundamental connection between the invariant element of inertial frame *proper-orientation*, $\mathrm{d}\theta$, and the invariant element of proper-time, $\mathrm{d}\tau$. In the most general circumstances, corresponding to compositions of boosts along different directions, both *non-exact* differential forms are simply proportional

$$\mathrm{d}\theta = \omega_{th}\,\mathrm{d}\tau \tag{3}$$

where $\mathrm{d}\tau$, is the element of proper-time of $K(\vec{v})$ (or of $K'(\vec{v'})$, at this order), and where, along a given worldline, ω_{th} is the instantaneous Thomas-Wigner rotation velocity.

From this "spatial side", the character of reality is manifest, and bears on Lorentz invariant quantities. The following is meant. Right after Thomas subtle discovery, it has become possible to find out a missing factor of $1/2$ (and not simply corrections of order v^2/c^2) reconciling some fine structure of alkalis doublets calculations, with the corresponding experimental results.

Today, accurate technologies, such as the *Global Positioning System* rely also on the soundness, persistence and reality of Thomas-Wigner rotations at any scale [3]. Likewise, in [13], new quantum mechanical phenomena are also presented in their relation to the Thomas-Wigner rotation. Indeed, the reason for such a universality is that Thomas-Wigner rotations are pure kinematical effects, and do not depend on the scale or dynamics of the situation considered.

3.3 The Proper-Time Line Functional

It matters to specify how are defined the proper-time lapses. They are defined by means of a *stratagem* first proposed by A. Einstein and since then, reformulated the same by others, like A. Schild for example [24]. The stratagem consists in: "*... imagining an infinity of inertial frames moving uniformly, relative to the laboratory frame, one of which instantaneously matching the velocity of the considered system, a twin, a clock, a particle...*" (23). Considering $\mathcal{C}$, a worldline of the spacetime manifold $\mathcal{M}$, the proper-time lapse along it is the line functional

$$\Delta(\mathcal{C};\mathcal{R}) = \int \mathrm{d}\tau_{\mathcal{C}/\mathcal{R}}(p), \quad \forall p \in \mathcal{C} \tag{4}$$

where $\mathcal{R}$ stands for some inertial "laboratory" frame of reference. $\mathcal{R}$ is arbitrary, but conveniently choosen in practice. Thanks to the above procedure, it is worth noticing that the proper-time line functional is a mathematically well defined quantity for any worldline. In particular, it is consistent, irrespective of the global spacetime manifold geometry. Gravitation/curvature, if any, must show up as an emergent or reconstructed effect, once admitted the equivalence principle (if gravitation is to be accounted for by general relativity, of course) [22]. Moreover, as noticed by A. Schild, since in (4), $\mathrm{d}\tau' s$ add up to $\mathrm{d}\tau' s$, no clock's synchronization difficulties are met along this procedure.

At the formal level, the stratagem finds a natural setting within the framework of differential geometry, in particular with the notions of tangent and cotangent fiber-bundles over a manifold. Eventually, contrarily to what can be read in the literature sometimes [23], accelerations are not erased by the procedure. This point will be made explicit in (17).

Considering $\mathcal{C}'$, another worldline with two points in common with $\mathcal{C}$, say 0 and ι, (4) will produce a differential aging result of

$$\Delta(\mathcal{C},\mathcal{C}';\mathcal{R}) = \int_0^\iota \mathrm{d}\tau_{\mathcal{C}/\mathcal{R}}(p) - \int_0^\iota \mathrm{d}\tau_{\mathcal{C}'/\mathcal{R}}(p') \equiv \delta T \tag{5}$$

As will be seen shortly, this result is non vanishing because of the path 4-velocity and 4-acceleration dependences which differentiate $\mathcal{C}$ from $\mathcal{C}'$ in the general case. It accounts for any acceptable twin paradox and associated asymmetry, in the following sense. Exchanging $\mathcal{C}'$ and $\mathcal{C}$ just amounts, as it should, to change δT into $-\delta T$. This is in contradistinction to the pure parallax effect of Sect. 3.1, where such an exchange, leaving the (non-Lorentz invariant) measure of *life-duration* unaffected, had motivated the original idea of a paradox.

4 Twin Paradox and Causality

The tight connection which relates the non trivial differential ageing phenomenon to the principle of causality is worth exploring. To do so, an important result must be recalled, and a few geometrical tools introduced.

4.1 A Theorem

The spacetime manifold $\mathcal{M}$ is assumed to be the connected $4D$ orientable affine Minkowski spacetime, M, endowed with a time-arrow. At any point p of the spacetime manifold M, the tangent space, $\mathcal{T}_p M$ is assumed to be the *vectorial* Minkowski spacetime $\mathbb{M}$.

A partial ordering can be defined on M, the *affine* Minkowski spacetime over the *vectorial* one, $\mathbb{M}$. This partial ordering expresses the principle of *causality*, attached to

(i) the existence of a finite speed limit concerning energy/information transfers, and
(ii) the existence of a time-arrow on M.

Writing $x < y$, if an *event* at x can influence another *event* at y, we will write [28],

$$x < y \quad \Longleftrightarrow \quad Q(y-x) \equiv (y_0 - x_0)^2 - (\vec{y} - \vec{x})^2 > 0, \quad \text{and} \quad y_0 - x_0 > 0 \tag{6}$$

Let f be a function, not even assumed to be linear or continuous, defining a one-to-one mapping of M into itself. If f and f^{-1} preserve the partial ordering (6), in the very sense that

$$\forall x, y \in M, \quad x < y \quad \Longrightarrow \quad f(x) < f(y) \quad \text{and} \quad f^{-1}(x) < f^{-1}(y) \tag{7}$$

then, f is said to be a *causal automorphism* of M.

Causal automorphisms of M form a group, $\mathcal{G}$, the *M-causality group*, and a theorem states that at 3 spatial dimensions, the Minkowski space causality group $\mathcal{G}$, coïncides with the inhomogeneous orthochronous Lorentz group, augmented with dilatations of M (multiplication by a scalar) [28].

4.2 A Few Geometrical Tools

We have assumed an *absolute* [7] affine Minkowski spacetime manifold, M, made out of event-points. Now, this one and absolute *History* can be reported to many different space *and* time projections, at many different points choosen as origins.

Note that the partial ordering (6) itself, relies on an implicit temporal/spatial separation of the possible coordinates defined on M. Now, M is a *spacetime* entity, whose projection on temporal *and* spatial dimensions is not given *a priori*. A *canonical* procedure exists, though, which, given a unit timelike vector u, allows to decompose any vector $v \in \mathbb{M}$ into the *direct sum* of two pieces

$$v = (v \cdot u)u \oplus (v - (v \cdot u)u) \tag{8}$$

These two pieces can be defined the temporal and spatial components of v relative to u, and the decomposition (8) is frequently given the name of *a splitting of* v. Concerning the twin problem, the following related aspects are worth putting forth.

(i) In Minkowski spacetime, timelike worldlines play a most important role because they have been proved to "encode" the geometrical, differential and topological structures of M [17]. This is why the space and time projections of M relative to the 4-velocity field of a timelike worldline are of special interest. And this is particularly so for the twin paradox under consideration.
(ii) In the latter situation in effect, the travelling twin goes along a timelike worldline $\mathcal{C} \in M$, image of a twice differentiable mapping $r(s)$ of an interval of $\mathbb{R}$, $[s_i, s_f]$, into M. To this timelike worldline are associated a 4-velocity field, $\dot{r}(s)$ and a 4-acceleration field, $\ddot{r}(s)$. This means that at each instant s, the traveller disposes of a space *and* time

$\dot{r}(s)$-projection of M, hereafter denoted by $M_{r(s)}$, which is the vectorial space $\mathbb{M}_{\dot{r}(s)}$ endowed with the point $r(s)$ choosen as the origin.

(iii) The vectorial space $\mathbb{M}_{\dot{r}(s)}$ is of course identical to $\mathbb{M}$, the vectorial Minkowski space, since both are made out of vectors that are geometrical frame-independent objects. However, vectors are not known *in themselves*, but solely through the projections which determine their components relative, here, to a given 4-velocity field, $\dot{r}(s)$. In other words, $\mathbb{M}$ and $\mathbb{M}_{\dot{r}(s)}$ are identical with respect to their content, and still define different *physical* spaces, because their commun content is captured by means of different space *and* time projections.

In the literature, past and recent, these remarks allow to pin a most frequently overlooked consequence. So long as $\mathbb{M}$ and $\mathbb{M}_{\dot{r}(s)}$ are not put into correspondence some way or other, there is no possible comparison of vectors belonging to $\mathbb{M}$ to vectors of $\mathbb{M}_{\dot{r}(s)}$ (in the recent years, for example, neglecting this fact has lead to another relativistic paradoxical case known as "the velocity addition paradox" (in [20]).

Such a correspondence is proposed now.

Let $\mathbb{M}_u$ and $\mathbb{M}_{u'}$ be two spaces tangent to M at points p and p' respectively. The space and time projections of $\mathbb{M}$ are performed along the unit 4-velocity vectors u at p, and u' at p' and they define the spaces $\mathbb{M}_u$ and $\mathbb{M}_{u'}$. We will here propose that a *natural correspondence* between the spaces $\mathbb{M}_u$ and $\mathbb{M}_{u'}$ be provided by the pure boost from u to u', whose geometrical, frame-independent expression reads [5, 20]

$$\mathbb{B}(u', u) = 1 - \frac{(u' + u) \otimes (u' + u)}{1 + u' \cdot u} + 2u' \otimes u \tag{9}$$

The scalar product of (9) is given by the metric $g = \mathrm{diag}\,(+1, -1, -1, -1)$, and the symbol $u' \otimes u$ stands for the linear mapping of $\mathbb{M}$ into itself,

$$\forall x, u, u' \in \mathbb{M}, \quad u' \otimes u : x \longmapsto u'(u \cdot x) \tag{10}$$

The linear bijection so established possesses several interesting properties. It preserves (1) the scalar product g, (2) the orientation of $\mathbb{M}$ and, (3) its time-arrow. Eventually, it allows to compare spatial 3-vectors which belong to different inertial spacetimes, and to define a notion of *physical equality* (and *parallelism*) for these vectors.

This point deserves a few words of explanation. Let $\mathbb{E}_u$ ($\mathbb{E}_{u'}$) be the simultaneity local space of an observer located at point $p(p')$ with 4-velocity $u(u')$. These 3-dimensional linear spacelike subspaces of $\mathbb{M}$ satisfy a definition, $\mathbb{E}_u = \{x \in \mathbb{M} |\ x \cdot u = 0\}$ which comes from the Einstein synchronization procedure. Considering $v \in \mathbb{E}_u$ and $w \in \mathbb{E}_{u'}$, v and w will be said to be *physically equal* if the relation $v = \mathbb{B}(u, u')w$ holds (on account of this relation of physical equality, it is worth pointing out that the velocity addition paradox alluded to above, finds an elegant and consistent solution (in [20]).

The physical equality so defined is reflexive, $\mathbb{B}(u', u)^{-1} = \mathbb{B}(u, u')$, but it is not transitive. Referred to the coordinate axes of a given inertial space, those of the "stayed home twin" for example, the Thomas-Wigner rotation is an expression of this non-transitivity. For any 3 non-coplanar 4-velocity vectors, u, u' and u'' (equivalent to 2 non-collinear relative 3-velocities), the composition of 3 successive boosts, without rotation, is not a boost without rotation ... but a rotation without boost!

$$\mathbb{B}(u, u'') \circ \mathbb{B}(u'', u') \circ \mathbb{B}(u', u) = \mathbb{R}_u(u', u'') \tag{11}$$

Qualifying the three vectors, u, u' and u'' so as to get the two relative velocities, $\vec{v}_{Ox}$ and $\mathrm{d}\vec{v}_{Oy}$ of Sect. 3.2, one can check that the above $\mathbb{R}_u(u', u'')$ just reproduces the Thomas-Wigner rotation of (2), while providing it with a rigourous derivation: in textbooks in effect, it is tacitly assumed that 3-vectors, elements of $\mathbb{E}_u$-spaces, undergo trivial parallel displacements, as if they were immersed in some larger ambient euclidean space.

Before using these elements, an important remark is in order.

However *natural* the isomorphism (9) may be, it does not proceed from first principles and would not provide a *canonical* correspondence between $\mathbb{M}_u$ and $\mathbb{M}_{u'}$ spaces [5, p. 261]. This is because the correspondence based on (9) integrates a peculiar convention of synchronization, the Einstein convention, when several others *look* possible [9]. But on the other hand, it has been recognized recently how inherent to the relativity theory the Einstein's convention must be regarded: to be really consistent, a change of convention should only be thought of within a full modification of the relativity theory itself [19].

4.3 Application to the Twins

For the sake of twin paradox, M, the Minkowski spacetime will be referred to the inertial stayed home twin space with corresponding constant 4-velocity field u_0: M-time is therefore u_0-time, and M-space, the u_0-space [20]. M is thus the point O-referred affine Minkowski spacetime over $\mathbb{M}_{u_0}$.

It is endowed with a pseudo-distance d, defined through the Lorentzian non degenerate quadratic form Q of (6),

$$\forall p, p' \in M, \quad d(p, p') = Q(p - p') \tag{12}$$

As advertised, the travelling twin history is accounted for by a mapping $r(s)$ of an interval $[s_i, s_f]$ into M, whose range is the twin's worldline $\mathcal{C} = \{r(s) \mid s \in [s_i, s_f] \subset \mathbb{R}\}$.

With $r(s_i) = O$, the mapping $r(s)$ satisfies relations which identify s to the traveller proper-time:

$$\forall s \in [s_i, s_f], \quad \frac{\mathrm{d}r(s)}{\mathrm{d}s} \equiv \dot{r}(s), \quad \dot{r}^2(s) \equiv \dot{r}(s)\cdot\dot{r}(s) \equiv g(\dot{r}(s), \dot{r}(s)) = Q(\dot{r}(s)) = 1 \tag{13}$$

The usual frame-independent relation of M-time, t, to s reads

$$\mathrm{d}t = u_0 \cdot \dot{r}(s)\, \mathrm{d}s \tag{14}$$

and besides $r(s_i) = O$, there is no loss of generality in completing the "initial data" of the travelling twin journey with the condition $\dot{r}(s_i) = u_0$ (this corresponds solely to a given experimental protocol).

In this geometrical setting, it is easy to see that the causal relations of M pass to $M_r(s)$, the instantaneous traveller's Minkowski spacetime. That is, if x and y are any two events of M such that the causal relation (6) holds, i.e. $x < y$, then the same relation holds also in $M_r(s)$.

The following proof can be given.

The difference of points y and x, elements of the affine space M, defines a vector of $\mathbb{M}$, $y - x$, that is a geometrical frame-independent object (point (iii) after (8)). Since the metric tensor g is constant over M, its length, $Q(y - x)$, does not depend on the point $r(s)$ where it is taken, and it is conserved.

Further, we denote by $\mathcal{I}^+$, the future-directed part of the light-cone at $r(s_i)$. By definition of $\mathcal{C}$, a continuous future-directed timelike worldline, for all $s \in [s_i, s_f]$, one has $\dot{r}(s) \in \mathcal{I}^+$.

Now, $\dot{r}(s_i)$ and $\dot{r}(s)$ are timelike 4-vectors, and so is $v = y - x$ in virtue of the hypothesis $x < y$, which tells also that $v \cdot \dot{r}(s_i) \equiv v^0(s_i) > 0$. Then, one can rely on the following Lemma: *two timelike vectors a and b belong to the same $\mathcal{I}^{\pm}$ part of the light-cone if and only if* $a \cdot b > 0$, in order to conclude that $v \cdot \dot{r}(s) \equiv v^0(s) > 0$, and complete the proof that the causal order of M passes to $M_r(s)$.

That is, for all $s \in [s_i, s_f]$, a causal isomorphism can be found, $\varphi_{s,s_i} : M_{r(s_i)} \to M_{r(s)}$, sending the $M_{r(s_i)}$ base-point, $r(s_i) = O$, onto the base-point of $M_{r(s)}$, and mapping the (O- and $\dot{r}(s_i)$ -referred) partial ordering (6) of $M_{r(s_i)}$, onto the ($r(s)$- and $\dot{r}(s)$ -referred) partial ordering (6) of $M_{r(s)}$,

$$\forall q \in M_{r(s_i)},\ \varphi_{s,s_i}(q) \in M_{r(s)}, \quad \varphi_{s,s_i}(q) - r(s) = \mathbb{L}_{s,s_i}(q - r(s_i)) \tag{15}$$

where, in virtue of Sect. 4.1 (and of the equivalence of active and passive points of view), $\mathbb{L}_{s,s_i}$ is an element of $\mathcal{L}_+^{\uparrow}$, the homogeneous orthochronous Lorentz group, or a dilatation, which, both, act on vectors of $\mathbb{M}_{\dot{r}(s_i)}$ ($\equiv \mathbb{M}$).

In view of the Einstein stratagem of Sect. 3.3, the $\mathbb{L}_{s,s_i}$ can be further specified, and to do so, it is crucial to make contact with the closely related mechanism of Sect. 3.2, the Thomas-Wigner rotations.

At base-point $r(s_i) = O$, one can attach a *tetrad* of orthonormal basis vectors spanning the vectorial Minkowski space $\mathbb{M}_{\dot{r}(s_i)} = \mathbb{M}$, the set $\{e_0(s_i) \equiv \dot{r}(s_i),\ e_j(s_i);\ \ j = 1, 2, 3\}$. If the mapping $r(s)$ is of class C^4, the tetrad can be constructed along the Serret-Frenet procedure which extracts the basis vectors out of the first four differentiations of $r(s)$ with respect to s. Identifying $e_0(s)$ with $\dot{r}(s)$, at all proper instant s, the tetrad of Serret-Frenet complies with the Einstein stratagem of Sect. 3.3 which requires the same identification. The other three spacelike vectors, $e_j(s)$, $j = 1, 2, 3$, can be physically realized as gyroscopes on $\mathcal{C}$, and formally thought of as *gyroscopic vectors on* $r(s)$ [21, 26].

For all s, in effect, one has by construction of the tetrad

$$(e_0(s) \equiv \dot{r}(s))^2 = 1, \qquad \dot{r}(s) \cdot e_j(s) = 0, \qquad e_i(s) \cdot e_j(s) = -\delta_{ij} \tag{16}$$

Since the affine Minkowski spacetimes $M_{r(s_i)}$ and $M_{r(s)}$ are mapped into each other by the causal isomorphism φ_{s,s_i}, the tetrad $\{\varphi_{s,s_i *}\ e_0(s_i),\ \varphi_{s,s_i *}\ e_j(s_i);\ \ j = 1, 2, 3\}$ spans the vectorial Minkowski space $\mathbb{M}_{\dot{r}(s)}$, where, in a standard notation, $\varphi_{s,s_i *}$ is the differential of the application φ_{s,s_i}, at $r(s_i)$. That is, $\varphi_{s,s_i *} = \mathbb{L}_{s,s_i}$, in view of (15).

When relations (16) are satisfied, the tetrad is said to be *Fermi-Walker transported* along $\mathcal{C}$, and for the tangent mapping $\varphi_{s,s_i *}$, one gets for all $s \in [s_i, s_f]$

$$\mathbb{L}_{s,s_i} = T \exp \int_{s_i}^{s} \mathrm{d}s'\ \dot{r}(s') \wedge \ddot{r}(s') \tag{17}$$

where the symbol $a \wedge b$ is a shortand standard notation for the antisymmetric product

$$\forall z \in \mathbb{M}, \quad (\dot{r} \wedge \ddot{r})z = \dot{r}(\ddot{r} \cdot z) - \ddot{r}(\dot{r} \cdot z) \tag{18}$$

and where the symbol T stands for a prescription of time-ordering, because taken at different times, Fermi-Walker operators, $\dot{r}(s') \wedge \ddot{r}(s')$, do not commute.

This can be seen as follows.

As (16) is satisfied, one can define a *compass*, a couple (r, e_j) of mappings, where e_j is a vector-valued function of $s \in [s_i, s_f]$ on the worldline r [21, 26]. For all $j = 1, 2, 3$, the stratagem of 3.3, makes of (r, e_j) a *locally inertial compass*, or, a gyrovector. This means that as vectors on r, the e_j's remain locally equal to themselves, preserving their own proper directions. Within the terms of Sect. 4.1, the physical equality of $e_j(s+\mathrm{d}s) \in \mathbb{M}_{\dot r(s+\mathrm{d}s)}$ and $e_j(s) \in \mathbb{M}_{\dot r(s)}$ reads $e_j(s+\mathrm{d}s) = \mathbb{B}(\dot r(s+\mathrm{d}s), \dot r(s))e_j(s)$. Iterating the latter relation from proper instant s_i to s, it is immediate to realize that the tangent mapping $\mathbb{L}_{s,s_i}$ is made out of a string of infinitesimal boosts ranging from s_i to s, along $\mathcal{C}$,

$$\begin{aligned}\mathbb{L}_{s,s_i} &= \mathbb{B}(\dot r(s), \dot r(s-\mathrm{d}s)) \circ \cdots \circ \mathbb{B}(\dot r(s_i+\mathrm{d}s), \dot r(s_i)) \\ &\simeq T \prod_{s'=s_i}^{s} \circ \big(1 + \mathrm{d}s'(\dot r(s') \wedge \ddot r(s'))\big)\end{aligned} \tag{19}$$

where the second equality is approximated at first order in the proper-time increment $\mathrm{d}s'$. Then, a first order differential equation is readily formed out of (19), for $\mathbb{L}_{s,s_i}$, whose solution at initial condition $\mathbb{L}_{s=s_i,s_i} = 1$, is given by (17).

In the proof just given, the all important point is that the physical equality of the e_j's holds in a local sense, and not globally. Note also that a correct identification of the tangent mapping makes it mandatory to take the full tetrad evolution into account: a relation such as $e_0(s) \equiv \dot r(s) = \mathbb{B}(\dot r(s), \dot r(s_i))\dot r(s_i)$, while being satisfied also by $\mathbb{L}_{s,s_i}$, does not extend to the other three spacelike basis vectors, the $e_j(s)$ at $j \neq 0$.

Now, we possess all of the elements that are necessary to a complete description of the twin's histories. To summarize, a causal series of M-automorphisms can be constructed, the $\{\varphi_{s,s_i}; s \in [s_i, s_f]\}$, which entails the travelling twin timelike worldline,

$$\{\varphi_{s,s_i}(r(s_i)) \mid s \in [s_i, s_f]\} = \{r(s) \mid s \in [s_i, s_f]\} = \mathcal{C} \tag{20}$$

the instantaneous, frame-independent relation of M-time to $M_{r(s)}$-time, expressed as

$$\mathrm{d}t = \dot r(s_i) \cdot \varphi_{s,s_i *}(\dot r(s_i))\, \mathrm{d}s \tag{21}$$

as well as the instantaneous Thomas-Wigner rotation of the spatial coordinate axes, the $e_j(s)$, with respect to the stayed home twin axes, provided that $e_0(s) = \dot r(s_i)$. This condition, in effect, is mandatory in order to have identical 3-dimensional spaces, and define meaningful rotations [21, 29],

$$e_j(s) = \varphi_{s,s_i *}(e_j(s_i)), \quad s \in [s_i, s_f],\ j = 1, 2, 3. \tag{22}$$

In the end, one and the same causal series, $\{\varphi_{s,s_i}; s \in [s_i, s_f]\}$, encodes not only the continuous timelike curve $\mathcal{C}$ itself (20), but also all of its local (differential) and global (integrated) characteristics:

- The fact that the twin's spaces may be found rotated with respect to each other, even though no torque has been met during the trip, (22).
- The non-trivial differential ageing phenomenon, by integration along $\mathcal{C}$ of the non-exact differential proper-time 1-form (21). Effectively, an equality of proper-time lapses, $t_f - t_i$ and $s_f - s_i$ does not hold in the general case where one has, if again $\dot r(s_f) = \dot r(s_i)$,

$$\int_{s_i}^{s_f} \mathrm{d}t(s) = t(s_f) - t(s_i) \equiv t_f - t_i = \oint_{s_i}^{s_f} \mathrm{d}s \dot r(s_i) \cdot \varphi_{s,s_i *}(\dot r(s_i)) \geq s_f - s_i \tag{23}$$

for those of the tangent automorphisms, $\varphi_{s,s_i *}$ that are in $\mathcal{L}_+^\uparrow$. If, instead, $\varphi_{s,s_i *}$ are *contractions*, i.e., dilatations by factors smaller than 1, then the opposite relation obviously results, of $t_f - t_i < s_f - s_i$, indicating that causality alone does not tell whom, of either twin, is ageing faster.

Note that (23) is more familiar if we keep in mind that the scalar product $\dot{r}(s_i) \cdot \varphi_{s,s_i *}(\dot{r}(s_i))$ can be thought of as the usual γ factor of $\sqrt{1-\vec{v}^2(s)/c^2}^{\,-1} \geq 1$. However, this sends us back to the *paradigm* of inertial frames of reference and their relative uniform velocities, whereas in the present geometrical context (23) enjoys a direct derivation where causality is the key argument.

The proof is as follows.

Causality is implemented on M through the partial ordering relation (6), that is, through the Lorentzian non-degenerate quadratic form Q, or, identically, the scalar product g. Now, because $\dot{r}(s_i)$ is timelike, the inverted Cauchy-Schwarz inequality holds,

$$\forall v \in \mathbb{M}, \quad g(\dot{r}(s_i), \dot{r}(s_i)) \times g(v, v) \leq g^2(\dot{r}(s_i), v) \tag{24}$$

where there is equality whenever v and $\dot{r}(s_i)$ are linearly dependent. Furthermore, since $\dot{r}(s_i)$ is of unit length ($\dot{r}^2(s_i) = 1$), for those of the tangent automorphisms $\varphi_{s,s_i *}$ that are in $\mathcal{L}_+^\uparrow$, one has $\dot{r}(s_i) \cdot \varphi_{s,s_i *}(\dot{r}(s_i)) \geq 1$ in view of (24), and the twin paradox (23) follows.

We can conclude this section with two remarks, the first of which on the role of acceleration, the second one, on two extra and implicit ingredients at the origin of the re-qualified twin paradox (23).

(i) In the twin paradox, in effect, accelerations have been the matter of long debates [9]. In (23), the integrand may be made explicit with the help of (15) and (17), and reads

$$\dot{r}(s_i) \cdot \left(T \exp \int_{s_i}^{s} \mathrm{d}s' \dot{r}(s') \wedge \ddot{r}(s') \right) \dot{r}(s_i) \tag{25}$$

Then, it appears clearly that if the 4-acceleration field of the travelling twin worldline is identically zero, then $t_f - t_i = s_f - s_i$, and there is no differential ageing whatsoever.

In Minkowski spacetime, a flat manifold, this point has long been understood as due to the unescapable necessity for the travelling twin to accelerate in order to come back and meet his brother again. A disymmetry shows up between the twin's worldlines, and a non-trivial differential ageing becomes possible.

(ii) Staring at (23), one may notice two implicit, essential ingredients entering the twin paradox recepee, with, first, in the right hand side, a *chronogeometrical axiom* according to which the geometrical elements of length divided by c, $\sqrt{\mathrm{d}s^2/c^2}$, do coincide with times recorded by (ideal) clocks, that is their proper-times, $\sqrt{\mathrm{d}\tau^2}$. And then, on the simpler left hand side, the (conventional) differential standard of time, taken out of *inertial motion*, and defined so as motion looks simple [22].

In effect, given any two timelike worldlines $\mathcal{C}$ and $\mathcal{C}'$, with two intersecting points in common, O and ι, a standard of time, inertial here, allows a consistent comparison of their respective lengths between O and ι. And provided the initial and final absolute 4-velocities coincide, one obtains an invariant well defined result. In (4) and (5), this point was meant by explicitly quoting the reference to some arbitrary inertial frame, $\mathcal{R}$. In the lack of such a standard of time, it seems impossible to *decide* which of either twin is ageing faster. More precisely, each twin can equally draw the same conclusion concerning his brother, and a definite motion must be choosen as a standard reference for time, so as to break a new symmetrical, still undefined case. This point could be worth analyzing in more details elsewhere, in view of its relations to other aspects of relativity theories.

5 Discussion

Since it was launched by P. Langevin in 1911 (and was indeed explicit in the Einstein's 1905 famous article), the twin paradox has been at the origin of more than 25,000 articles in the literature. The paradox proper framework was soon identified with general relativity theory because of accelerations to be considered along a twin worldline, at least. This point of view was adopted by Einstein and supported by M. Planck. However, it has long been recognized to be at fault, for both theoretical and experimental reasons [22]. In particular, as long known by particle physicists, accelerations can be consistently dealt with in flat spacetime manifolds, and should no way be mistaken for gravitation [7, 22].

In this article, our intention has been to look for the principle at the origin of a so counter-intuitive, but established fact as "the non-trivial differential ageing phenomenon". And to this end, it was certainly appropriate to look for such a principle in the simpler structure where the phenomenon is manifest, that is over the local scale of a Minkowski spacetime manifold. It is worth noticing that long ago, A. Schild had also developed serious arguments in favour of this point of view [24].

At first, the twin paradox has been re-qualified into the fact of path-functional dependence of proper-time lapses, while preserving the name so as to keep in touch with the original terminology.

In the end, we have seen that, given a chronogeometrical hypothesis and a definite convention of time, then, one and the same series of M-causal automorphisms contained all of the local and global physical properties of the paradoxical case. Not only the non-trivial differential ageing phenomenon itself, but also the somewhat related Thomas-Wigner rotations. These are the ingredients of the re-qualified twin paradox.

Ten years ago, causality was advocated to provide a global constraint on the possible twins histories, labelled, each, by some experimental/theoretical synchronization device [9]. Here, more than an overall constraint on the twin's histories, one can see how the sole principle of causality stands at the very source of the twin paradox. That is, how preservation of causality along continuous timelike worldlines necessarily involves an explicit dependence of proper-time lapses on the paths 4-velocity and 4-acceleration fields.

In this respect, the famous paradox may be looked upon in analogy with those situations encountered in Mathematics, where unquestionable axioms can sometimes generate counter-intuitive … if not "paradoxical" consequences [12].

Now it is obvious that contrarily to the *Galilean* case, the causality dealt with in this article comes from the existence of a finite speed limit to the energy and information transfers on M. So that eventually, the twin paradox (and beyond, the whole special relativity theory itself) should be understood as a consequence of the existence a finite speed limit on M.

This point of view complies with that of Malament 1986, whose considerations start from general relativity. In particular the relation of general relativity to Newtonian physics is investigated, and the all important difference between the two theories comes out to be that the maximal speed goes from finite to infinite. A noticeable by-product of the analysis is that in the limiting process, the proper-time non-exact differential becomes exact, and that the twin paradox disappears accordingly.

This begs the question of knowing how to set up the existence of such a limit. As well known, such a limit was originally postulated by Einstein together with the principle of relativity; the experimental support being the famous Michelson-Morley experiment. Later on, the sole principle of relativity, in particular its group structure requirements, revealed sufficient a constraint to infer the existence of a universal constant with the dimensions of a velocity [8, 15]. The latter has been identified with the velocity of electromagnetic waves propagation in the vacuum, but this identification is not mandatory [14]. Along this route though, in order to develop the full special relativity theory, one has to consider extra assumptions like linearity, spacetime homogeneity, space isotropy [8, 15], some of them hard to support nowadays (E. Bois, private communication).

As clearly stated by Zeeman's theorem, such is not the case if one posits a finite speed limit in M. In terms of postulates, this may be the more economic way to establish and develop the relativity formalisms. This appears the more so as, starting from general relativity, all of the so peculiar relativistic features fade away as the infinite speed limit is taken [18]. The issue of a finite speed limit is therefore of utmost importance in physics, and it is a formidable challenge, from experimental, theoretical and epistemological standpoints. However, one must acknowledge that this position is not shared by all authors. For example, in a recent presentation of relativity theory [25], it is explained that "…(*the speed of*) *light has nothing to do with the essence of relativity*", and a construction is proposed to display how "natural" is the Minkowskian character of our spacetime environment! We would rather think of it as a somewhat involved (re-)construction, though, and simply remind that when he discovered his theory under the Minkowskian formulation, A. Einstein had experienced some difficulties at recognizing it!

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